Converting between power law and hyperbolic representations of stiffness decay with strain

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ABSTRACT
Repeatable data for shear modulus is readily obtained from small cycles undertaken as part of a cavity expansion test. In all soils these data follow a hysteretic path that a high resolution pressuremeter can capture and use to describe the stiffness degradation with strain characteristics. This response is normally modelled as a power law. The standard cycle is not expected to see shear strains below 0.01% and hence does not measure the elastic shear modulus, G_{max}, found at shear strains nearer 0.001%.

By contrast, laboratory testing of small strain stiffness is normally represented by a hyperbolic function where G_{max} defines the starting plateau, and in this form is readily incorporated into design software. It is always important that laboratory and insitu data are compared and if possible aligned. This paper considers potential methods for translating power law data into hyperbolic relationships.

RÉSUMÉ
Des données répétables pour le module de cisaillement sont facilement obtenues à partir de petits cycles entrepris dans le cadre d'un test d'expansion de cavité. Dans tous les sols, ces données suivent un chemin hystérétique qu'un pressiomètre à haute résolution peut capturer et utiliser pour décrire la dégradation de la rigidité avec des caractéristiques de déformation. Cette réponse est normalement modélisée par une loi de puissance. Le cycle standard ne devrait pas voir des déformations de cisaillement inférieures à 0,01 % et ne mesure donc pas le module de cisaillement élastique, Gmax, trouvé à des déformations de cisaillement plus proches de 0,001 %.
En revanche, les essais en laboratoire de rigidité à faible déformation sont normalement représentés par une fonction hyperbolique où Gmax définit le plateau de départ et, sous cette forme, est facilement intégré dans le logiciel de conception. Il est toujours important que les données de laboratoire et in situ soient comparées et si possible alignées. Cet article examine les méthodes potentielles pour traduire les données de loi de puissance en relations hyperboliques.

1 INTRODUCTION

Cavity expansion testing is acknowledged to be a reliable means for determining ground stiffness. Such a test conducted with a high resolution pressuremeter consists of an expansion phase, where the ground is loaded, a contraction phase where the ground is unloaded and several unload/reload cycles where the applied stress is reversed. An example of such a test carried out with a pre-bored pressuremeter is shown in Figure 1.

For a prebored test, the pressuremeter is placed in a slightly oversized pocket formed by conventional drilling tools. Although the initial slope of the field curve could be used for stiffness interpretation it is affected by the pocket forming process and subsequent soil relaxation

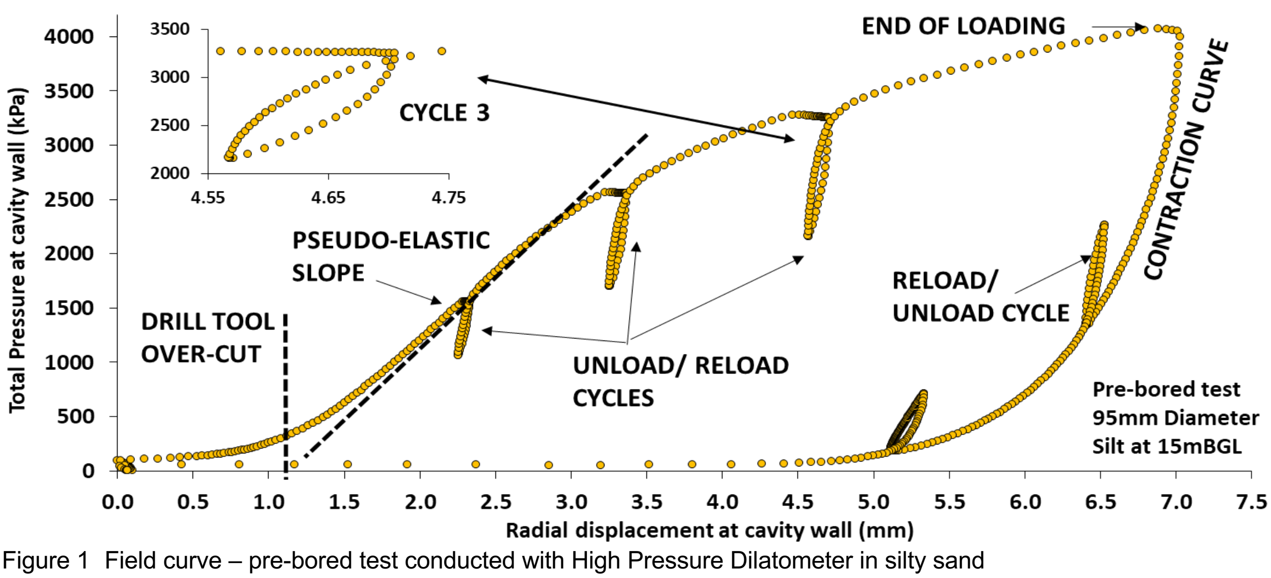

Figure 1 Field curve – pre-bored test conducted with High Pressure Dilatometer in silty sand

Consistent and repeatable data for shear stiffness can be obtained from small cycles of unloading and reloading (Hughes, 1982). The response of these cycles is driven by the far field pre-yield properties of the ground.

This can be demonstrated by the similarity of successive cycles. Any cavity expansion test has the potential to produce high quality stiffness data regardless of the damage caused to immediately surrounding material by the insertion process.

With high resolution pressuremeters, it is apparent that the path followed during these cycles is not linear, as might be assumed from low resolution measurements, but is hysteretic in all soils. Muir Wood (1990) and Jardine (1992) examine the response of these cycles and propose ways of obtaining stiffness degradation data from them. Bolton & Whittle (1999) give a straightforward method based on a power law and it is now customary to apply this approach to the standard pressuremeter test.

Small strain stiffness laboratory testing attempts to determine the elastic shear modulus G_{max} and then follow the subsequent degradation as shear strain increases. This encourages the use of a hyperbolic function to describe the overall response. By contrast, the smallest observable change in the pressuremeter test generally means that the current stiffness is already at an arbitrary point on the degradation curve (Figure 2) and hence a power law approach works well. The purpose of this paper is to show that the two approaches can be reconciled.

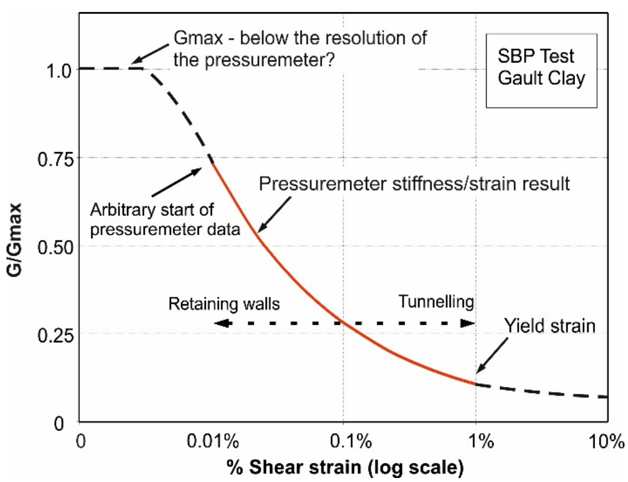

Figure 2 Normalised stiffness/strain response of a characteristic SBP test in clay

2 THE POWER LAW

To describe the non-linear stiffness/strain response, Bolton & Whittle give the following relationship:

$$G_s = \alpha\gamma^{\beta-1} \quad [1]$$

Where G_s is secant shear modulus at plane shear strain γ, α is the shear stress constant and β is the exponent of non-linearity. β takes a value between 0.5 and 1, where 1 is linear elastic.

Either the unload or reload part of cycle can be used, but in practice the second half of the cycle (data obtained after the turnaround point) is least likely to be affected by creep effects from the ground or frictional errors introduced by the measurement system. By fitting reloading data with a power curve, α and β are found. The correlation is generally better than 0.99. This method is valid for measured shear strains in the range between 10^{-4} (0.01%) and the yield strain γ_f (the strain when the full strength of the material is mobilised). This is typically 10^{-2} or 1% for over-consolidated clays.

The shear modulus at 10^{-4} shear strain is likely to underestimate the elastic shear modulus (G_{max}). Although the curve can be projected back beyond 10^{-4} it is necessary to know the threshold strain (γ_{elas}) at which stiffness decay commences. Without this, power curve modelling predicts infinite stiffness at zero strain. The elastic threshold strain is typically in the order of 10^{-5}.

The pressuremeter lower measurement limit of 10^{-4} is due to a variety of reasons including:

- the limitations of the electro-mechanical measurement system.
- the possible influence of ground creep
- the fine pressure control challenges.

In principle the resolution of the displacement measuring system is sufficient to see G_{max} (shear moduli in excess of 1GPa are routinely measured in rock) but every change in direction is affected by a small loss due to friction and dynamic ground effects, loosely termed 'creep'. Additionally, stiffness decay will commence after a tiny proportion of the available strength is mobilised, typically 1%. Hence if a material has a shear strength of 100kPa it means that all G_{max} data cease after a change of 1kPa. To make a credible assessment with the pressuremeter a minimum of 5 steps of 0.2kPa change would be required.

In a triaxial chamber where all elements are seeing the same shear stress it is relatively straightforward to arrange for this fine degree of stress control. However, in the cavity expansion test the current shear stress seen by an element of soil depends on its radius relative to the cavity wall. The field curve is the integration of these infinite stress states.

Consequently there will always be a propensity for the ground to creep as it attempts to reconcile the different rates of strain whenever the expansion process is paused or reversed.

Figure 2 uses data from a self bored pressuremeter test in Gault Clay, with the part of the curve in red indicating the information that is provided routinely by the pressuremeter unload/reload cycle. This is the range of strain applicable to most geotechnical design and hence it is desirable that this be directly measured insitu.

3 STIFFNESS AS A FUNCTION OF STRENGTH

The stiffness decay process starts at the elastic threshold strain and terminates at the yield strain required to initiate full plasticity. This is largely recoverable. It is convenient to present stiffness decay against the fraction of mobilised shear stress. This can be written as follows:

$$G_s^n = \alpha[n\, c_u/\alpha]^{(\beta-1)/\beta} \quad [2]$$

where $0 < n \leq 1$ and is the proportion of strength used. c_u is undrained strength and for the drained case will be τ_f (the shear stress required for first yield).

This is based on Bolton & Whittle, which can be rearranged to give shear modulus in terms of shear stress rather than shear strain using the power law definition.

High resolution pressuremeters can make field measurement adequate for n values between 0.05 to 1. This resolution is not enough to see the elastic shear modulus G_{max}, since it is likely to degrade after only 1% of the available strength has been mobilized ($n = 0.01$). Similarly, it is difficult to recognise the elastic strain threshold, γ_{elas} from measurements made during a cavity expansion process. However, if either of the parameters are known, then the power law can be arranged to describe the full decay response.

4 CALCULATING ELASTIC SHEAR MODULUS

The cavity expansion test is particularly good at identifying the yield condition. If both the hyperbolic and power law descriptions are assumed equivalent, then it can be inferred from Cao et al (2002) that there is a relationship connecting G_y, the shear modulus at first yield, to the elastic shear modulus G_{max}:

$$G_{max} = G_y\, Exp[1/\beta\,] \quad [3]$$

If measured values of β are used then Eq. 3 tends to give conservative values for G_{max}. However, in all soils that can be described as non-linear, at the point of measurement, β represents the particle size and shape at the microscale. Differential stress will lead to asperity removal and increasing uniformity. Ultimately, β will tend towards 0.5 as elements approach spherical form. If it is assumed that β is 0.5 when using Eq. 3 then it follows that:

$$G_{max} = 7.38 G_y \quad [4]$$

This is speculative. However, if justified, it leads to the conclusion that Eq. 4 gives the maximum that G_{max} can be. Once G_{max} is known, then Eq. 1 or Eq. 2 can be re-arranged to find the threshold strain γ_{elas}.

5 THE HYPERBOLIC APPROACH

A hyperbolic function is the conventional way of representing stiffness decay. G_{max} is used as the primary input to define the starting plateau. A simple hyperbolic arrangement (e.g. Hardin & Drnevich, 1972) is mathematically convenient but is a poor representation of the curvier response of real soils. Oztoprak & Bolton (2013) review some variations and suggest the following, based on Darendeli (2001):

$$\left(\frac{G_s}{G_{max}}\right) = 1\Big/\left[1 + \left(\frac{\gamma - \gamma_e}{\gamma_{ref}}\right)^m\right] \quad [5]$$

where γ_{ref} is a reference shear strain when $\frac{G}{G_{max}} = 0.5$ and m controls the curvature of decay (Oztoprak & Bolton denote this α but here m is used to avoid notation duplication). Oztoprak & Bolton use Eq.5 to model the decay response of a wide range of tests conducted in sand:

The primary difference between Eq. 5 and earlier representations is the curvature exponent. This gives a more realistic representation of the material response but at the expense of an additional unknown.

The exponent is the only sand specific element of Eq. 5. Oztoprak & Bolton found that $m = 0.88$ gave the best average fit to their database of sand tests. The curvature is potentially related to particle size and will therefore vary with soil type, analogous to the way β operates in the power law method.

5.1 Application to pressuremeter tests

For Oztoprak & Bolton's analysis, G_{max} is known and stiffness decay is the predicted element. For the pressuremeter test the majority of the stiffness decay relationship is measured from unload/reload cycles, however, G_{max} is not. An iterative approach can be applied where successive estimates of G_{max} can be used in Eq. 5 to find a hyperbolic data set that reproduces the pressuremeter decay data (Figure 3). Here, the strain axis is proportion of mobilised shear strain. It starts from zero and ends at the yield strain γ_f.

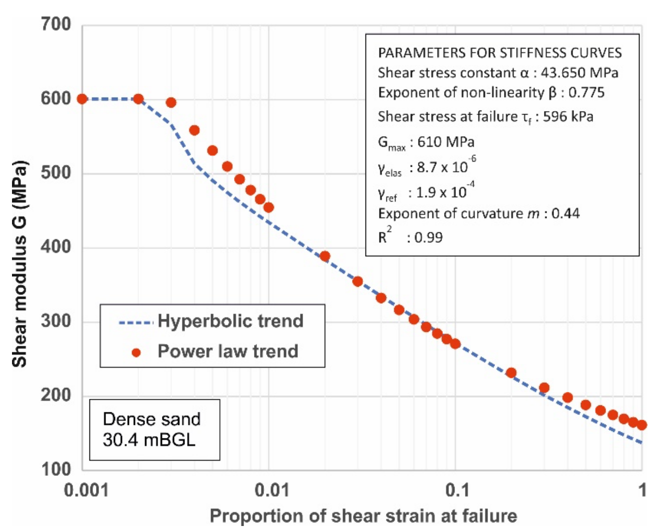

Figure 3 Matching hyperbolic and power curves

The plane shear strain at failure for the undrained case is given by Eq. 6. This can be arranged to find the mobilised shear stress for any shear strain between zero and γ_f. In Figure 3, this can be used to define the horizontal axis.

$$\gamma_f = \left[\frac{c_u}{\alpha}\right]^{1/\beta} \quad [6]$$

For the drained case, c_u can be substituted with τ_f (the shear stress at first yield). The shear stress constant α must be appropriate for the effective stress level at failure.

There are several methods (Bellotti et al, 1989, Whittle & Liu, 2013) for adjusting stiffness and shear stress constant for stress level. It is common for the first valid unload reload cycle to be used as an initial approximation. The first cycle is often initiated sufficiently close to the insitu stress state for its power law constant and exponent to be used directly.

Figure 4 is an example of the potential difficulty. The plot consists of 5 cycles taken from a self bored test in dense sand. The power law results can be presented as continuous lines. It is also possible to extract stiffness decay data directly from the experimental measurements, and these are the points on the same plot. By of plotting both data sets in this way the yield strain can be easily identified, where measured data separates from calculated trends. If the cycles are large enough then it is common to slightly exceed the shear stress that applied when the cycle was initiated and the break is therefore the material yield strain.

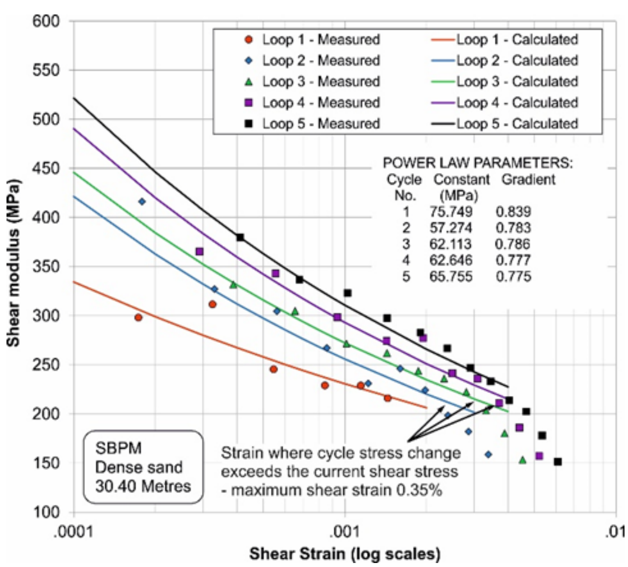

Figure 4 Stiffness degradation curves from a drained test in dense sand

5.2 Summary

The hyperbolic iterative approach to determining G_{max} with a pressuremeter has been applied to a number of sites and materials of varying permeability grading from clays to sands. The correlation coefficient between data sets predicted by the power law parameters and the hyperbolic parameters is typically 0.99. The iterative procedure stops when the correlation is at a maximum.

As suggested above there is an inverse relationship between m and β. Based on tests examined to date, the following seems to fit most cases:

$$m = 1.5(1-\beta) \quad [7]$$

There is evidence that where m requires additional manipulation, at least one of the other parameters is likely to have been wrongly assessed.

6 MEASURING $\boldsymbol{G_{max}}$ WITH THE PRESSUREMETER

It is clearly desirable to measure G_{max} directly rather than rely on calculation or iteration. This is theoretically achievable with an appropriate procedure, despite the difficulties associated with the mechanical limits of the measurement system.

'Cycles within cycles' is one possible procedure. Figure 5 is a sketch of the concept. Having taken the unloading part of a cycle, the reloading phase would be only 50% of the amplitude of the initial unload. A further unloading down to the same lower limit of the cycle would then be made. The process repeated until the change of stress is sufficiently small to give a near linear response. The slope of the linear response would give G_{max} directly. Once G_{max} is obtained, the final reloading would return the cycle to the original loading curve.

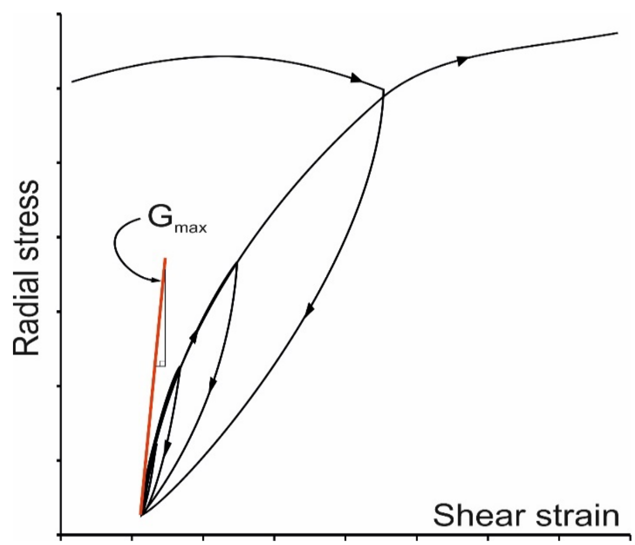

Figure 5 Cycles within cycles to identify G_{max}

Any friction at the turn-around point would be the same for all cycles within cycles and hence recognisable. Any creep is a fixed percentage of the process prior to a reversal of direction. Reducing the stress amplitude of successive cycles, makes the impact of creep negligible.

When considering soils, creep is a characteristic of unload/reload cycles, in all soil types. The standard unload/reload cycle applies shear strain greater than the elastic threshold, and under those conditions all soils creep due to micro fracturing and particle re-arrangement. Creep is a generic term covering a multiplicity of actions and is can be an imprecise definition. Hence clays will in general appear to creep more than sands whenever the loading path is reversed simply because the response tends to be undrained and any pause initiates loss of excess pore water pressure. Technically this is not creep, however the consequence is similar: it introduces a time dependent deformation for no increase in the applied stress. In addition, there are rate effects; even a slow pressuremeter test is rapid compared to the process of laboratory testing. However, if multiple cycles of ever-reducing size are carried out at the same point in the test, then creep ceases to be a significant issue.

The primary challenge is not related to displacement resolution. Pressuremeter tests in competent rock require the accurate measurement of sub μm displacements to give data for stiffness moduli. The problem is applying pressure increments in small enough steps and reading the consequent response at a fast enough rate.

One possible solution for this is the use of a load cell pressuremeter (LCPM) system as described by Hughes and Whittle (2023). The LCPM is capable of measuring G_{max} as a fortuitus by-product of making measurements of the insitu lateral stress. Conceptually, the LCPM is the reverse of the expansion pressuremeter. The stress in the ground bears on the surface of a load cell, causing it to deflect inwards. These sub μm inward movements are used as feedback for a control system that raises the internal pressure to match the external stress and hence restore the load cell to a zero strain condition. This process takes some time. The readings made whilst this process is occurring can be interpreted as stiffness information. However, the LCPM is essentially a research tool and is seldom used in a wider commercial sense.

6.1 Field tests

As part of the 1992 ground investigations for the London Crossrail project, cavity expansion tests using a self-bored pressuremeter (SBP) were carried out with unusually small unload/reload cycles. The purpose was to obtain stiffness data at much smaller strain levels than was then customary. To achieve this, all cycles were controlled by a semi-automatic pressurisation system able to make relatively small pressure changes.

If the data from these cycles are examined then it is possible to see that following the turnaround point in the cycle, very small strain data are available that may be representative of the elastic shear modulus. Figure 6 is an example of one such cycle.

This cycle mobilises less than 15% of the available strength. The conventional approach, as in Figure 1, utilizes at least 50% of the available strength. Consequently, much of the hysteresis or curvature of the conventional cycle is not apparent in the very small cycle. The initial linear portion following cycle turnaround has been used to quantify G_{max}. There is good agreement between this and G_{max} calculated using Eq. 3. It should be noted that the G_{max} estimation from Figure 6 is based on only three experimental data points. The value obtained should be considered as proof of concept rather than reliable measurement.

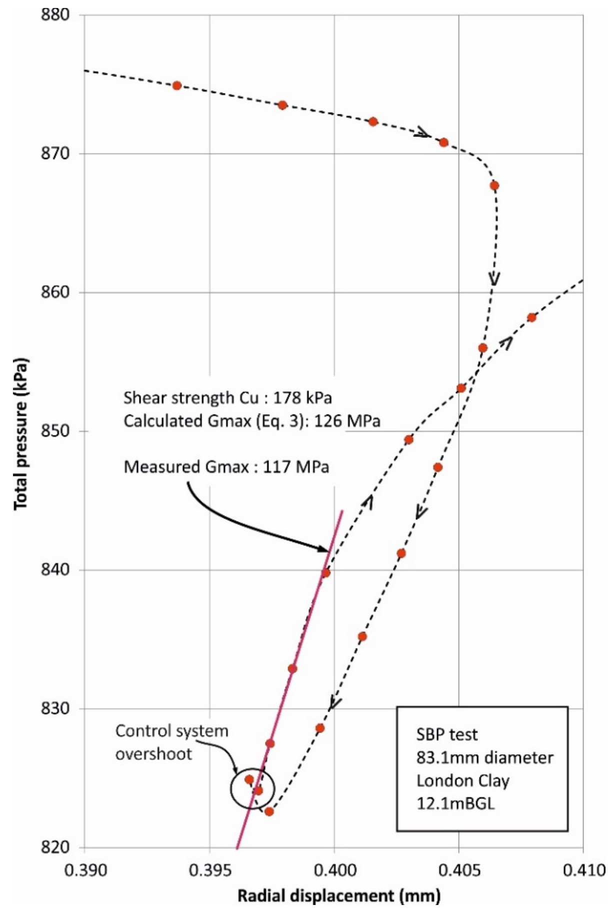

Figure 6 A small unload reload cycle examined for G_{max}

Figure 7 gives the results from 30m of testing in one borehole from Crossrail. Data are shown for shear modulus at 0.01% shear strain; this is a reliable measurement but obtained at too large a strain to be G_{max}. The results for G_{max} derived using Eq. 3 and from the measurement technique displayed in Figure 6 are plotted alongside. These show a stiffer response and seem plausible.

These tests were not designed for the purpose for which they are being used here. However, the comparison between the measured and calculated G_{max} trends is encouragingly close.

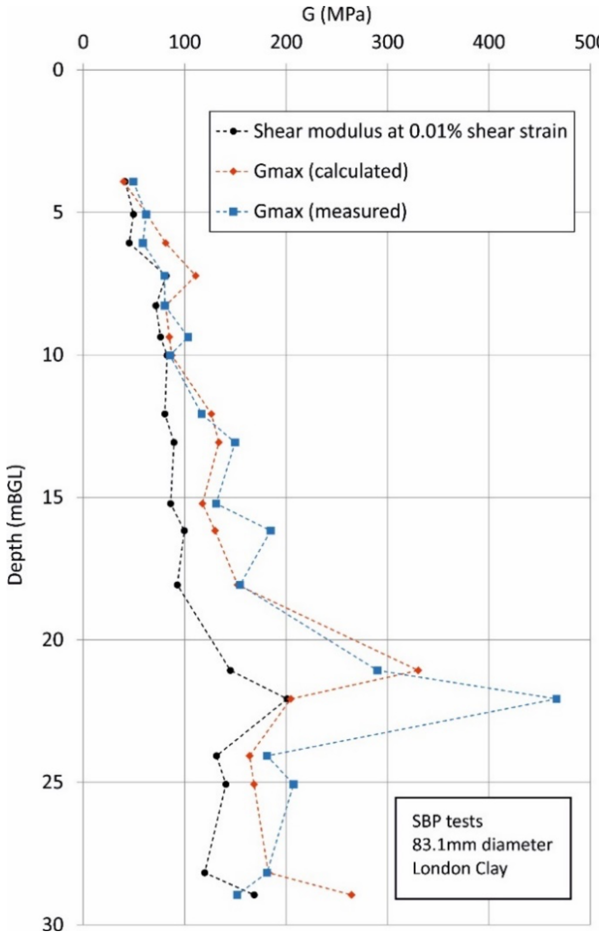

Figure 7 Deriving G_{max} from SBP tests in London Clay

7 CONCLUSIONS AND CLOSING REMARKS

Laboratory based small strain stiffness testing begins by applying tiny increments of stress that, in the ideal case, allow both the elastic modulus and elastic threshold strain to be identified. A cavity expansion test undertaken with a high resolution pressuremeter uses unload/reload cycles to determine stiffness characteristics. These cycles are concerned with strain levels greater than the elastic threshold but give modulus parameters representative of those applicable to engineering problems. As the start condition is undefined, it is reasonable to use a power law to quantify the response. No empiricism is required to produce the pressuremeter trends and there are sound reasons for considering the power law to be a better representation of the ground response than a hyperbolic function. However, power law parameters are not easily incorporated into design packages such as FLAC or PLAXIS.

This paper has described methods for translating between power law parameters and an equivalent hyperbolic function. To do this it is necessary to identify the elastic shear modulus G_{max}, which may be done in at least three ways:

- By iteration to find the value that gives the highest correlation coefficient to the experimental data. (Eq. 5)
- By using relationships that assume that the elastic threshold and yield strains are related. (Eq. 3)
- By direct measurement, using modified unload/reload cycles. (Section 6)

In practice, it is likely that all 3 methods would be deployed to find a consensus. For direct measurement to be successful it is necessary that the steps of pressure are kept very small. This will require an automatic control system. The reading rate will also need to be faster than commonly used commercial equipment currently provide; whilst maintaining the principle of quiet data over quantity of points.

The cycle procedure itself will need to be modified to minimise the influence of creep and mechanical imperfections if the direct measurement of G_{max} is intended. Within a test with 3 or more unload/reload cycles, it should be possible to arrange that one cycle could be prioritized for extracting small strain stiffness data.

Although most of the examples have come from undrained tests the suggested methods are applicable to the drained case. However, unload/reload data from the same test level will be influenced by variations in the mean effective stress and an appropriate adjustment for this would need to be introduced.

Used conventionally the pressuremeter gives parameters for G_{HH}, the lateral stiffness for a lateral loading. Comparisons with laboratory and other methods need to take account of this. G_{HH} often tends to be the orientation and loading direction giving the highest stiffness values.

It must be emphasised that the approaches outlined here are only applicable to high resolution instruments which make direct measurements of the radial movements of the cavity wall. It would be unreasonable to expect this level of discrimination from a volume measuring system.

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